JOURNAL OF ALGEBRA **85**, 287–302 (1983)

Efficient Generation of Ideals in Polynomial Rings

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Communicated by D. Buchsbaum

Received October 8, 1981

1. INTRODUCTION

Let R denote a commutative Noetherian ring with identity. If M is a finitely generated R-module we let $\mu(M)$ denote the cardinality of a minimal generating set for M. The conormal bundle of an ideal I in a ring R is the group I/I^2 viewed as an R/I-module. It has been observed that the algebraic properties of this module are closely connected with those of the ideal I. For instance, the fact that $\mu(I/I^2) \leqslant \mu(I) \leqslant \mu(I/I^2) + 1$ (which is easy to prove) shows this to be the case for the number of generators. Which of the above inequalities is an equality then becomes the question of interest. In this paper we are concerned with the attaining of equality with the lower bound for ideals in polynomial rings with coefficients in a regular local ring. In the rest of this introduction let A denote a regular local ring of dimension d and let $R = A[T_1,..., T_n]$ be a polynomial ring. We prove the following results.

(1) Suppose A is a formal power series ring over a field. Let I be an ideal in R. If $\mathrm{ht}(I) > \min\{d, n\}$ and $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$.

(2) Suppose that the residue field L of A is infinite and that A contains a field K such that A is a localization of an affine K-algebra and L is a finite separable extension of K. Let I be an ideal in R. If $\mathrm{ht}(I) > \min\{d, n+2\}$ and $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$. Further, in case I is a prime ideal in R such that $\mathrm{ht}(I) > \min\{d, n+1\}$ and $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$.

(3) Let $(D, (\pi))$ be a discrete valuation ring with infinite residue field. Suppose A is the local ring of $D[X_1,..., X_{d-1}]$ at the maximal ideal $(\pi, X_1,..., X_{d-1})$. Then every maximal ideal in R is a complete intersection. The results (1) and (2) generalize the following theorem of Mohan Kumar [13].

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0021-8693/83 $3.00

Mohan Kumar's Theorem. *Let k be a field and let $R = k[T_1, \ldots, T_n]$. If I is an ideal in R with $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$.*

As applications of our results we obtain that if R is a regular polynomial ring of the types described in (1) and (2) above, then maximal ideals in R are complete intersections. This settles a few important cases of the following conjecture due to Davis and Geramita.

Conjecture. Let $R = A[T_1, \ldots, T_n]$ be a polynomial ring in n variables, where A is a regular local ring of dimension d. Then maximal ideals in R are complete intersections.

It was primarily in this setting that our investigations began and the results of this paper took shape. The result (3) above cites some other instances where the conjecture is true. In [3] it was shown that if $n > 1$ then the conjecture is valid. Also, if $\dim(A) \leqslant 2$ then the conjecture is true as was the content of [4].

Bhatwadekar [2] has also independently proved that the conjecture is true when A is a formal power series ring over a field and also when A is a localization of an affine algebra over an infinite perfect field.

The methods to establish our results are via projective modules. The idea behind our proofs is to obtain projective R-modules of the "desired" ranks to map onto the ideals under consideration. We know that when R is a regular polynomial ring of the types mentioned in (1), (2) and (3) above, then all finitely generated projective R-modules are free [10, 11, 15]. We first record Mohan Kumar's theorem in a general form, Theorem 1.2. Then we axiomatise techniques related to those which were used by Lindel in [8–10] to prove his extensions of the Quillen–Suslin results. We divide this paper into three sections. In Section 1 we develop the techniques and establish the preliminary results needed for the main theorems. The main theorems are proved in Section 2. Section 3 is devoted to some of the applications of our results.

1. Preliminary Results

In this section we prove some preliminary results needed to establish the main theorems. All rings we consider are commutative and Noetherian with identity. By the dimension of a ring we mean the Krull dimension and we shall be concerned with only rings of finite dimension. If R is a ring and I is an ideal in R, then we shall use $\mathrm{ht}(I)$ and $\dim(I)$ to denote the height of I and the dimension of the ring R/I, respectively.

Definition 1.1. Let R be a ring and let M be a finitely generated R-

module. We define $\mu(M)$ to be the least number of elements in M required to generate M as an R-module.

As mentioned in the Introduction we record Mohan Kumar's theorem in the general form. The proof can be found in [6, Theorem 4].

THEOREM 1.2. *Let $R = A[T_1,..., T_n]$ be a polynomial ring over a ring A. Suppose I is an ideal in R which contains a polynomial monic in one of the variables. If $\mu(I/I^2) \geqslant \dim(I) + 2$, then I is the homomorphic image of a projective R-module P of rank equal to $\mu(I/I^2)$.*

Let R be a ring and R_1 be a subring of R. Let $f \neq 0$ be an element in R_1 such that f is a non-zero divisor in R. We shall say that $R_1 \subset R$ is an *analytic isomorphism along f* if $R_1/(f) \simeq R/(f)$ or equivalently $R = R_1 + fR$ with $fR \cap R_1 = fR_1$. We shall be using either of these formulations as convenient. It follows easily that if $R_1 \subset R$ is an analytic isomorphism along f then $R_1 \subset R$ is an analytic isomorphism along f^n as well for any positive integer n.

The following is the key proposition which makes it possible for us to use Theorem 1.2 in establishing our main results.

PROPOSITION 1.3. *Let R_1 be a subring of a ring R. If I is an ideal in R let $I_1 = I \cap R_1$. Suppose there is an element f in I_1 such that $R_1 \subset R$ is an analytic isomorphism along f. Then:*

(1) $R_1/I_1 \simeq R/I$,

(2) $I = I_1 R$,

(3) $\mu(I_1/I_1^2) = \mu(I/I^2)$.

Hence $\mu(I_1) = \mu(I_1/I_1^2) \Rightarrow \mu(I) = \mu(I/I^2)$.

Proof.

(1) Follows immediately.

(2) As $I = I_1 + fR$ and $f \in I_1$ it is clear that $I = I_1 R$.

(3) Replacing f by f^2 we may assume that $R_1 \subset R$ is an analytic isomorphism along f and $f \in I_1^2$. Then $R_1/I_1^2 \simeq R/I^2$. Now it is easily checked that I_1/I_1^2 and I/I^2 are isomorphic as R_1/I_1^2 or R/I^2-modules. Hence $\mu(I_1/I_1^2) = \mu(I/I^2)$.

Further, from (2) we have $\mu(I) \leqslant \mu(I_1)$. If $\mu(I_1) = \mu(I_1/I_1^2)$ then $\mu(I) \leqslant \mu(I_1) = \mu(I_1/I_1^2) = \mu(I/I^2) \leqslant \mu(I)$ and so $\mu(I) = \mu(I/I^2)$.

We now study some properties of polynomial rings.

DEFINITION 1.4. Let $(A, \mathfrak{m})$ be a local ring. We say that a monic polynomial f in $A[T]$ is a *Weierstrass polynomial* of degree n if $f = T^n + a_1 T^{n-1} + \cdots + a_n$, with $a_i \in m$ for $i = 1, 2, \ldots, n$.

PROPOSITION 1.5. *Let (A, m) be a local ring and $R = A[T]$ be a polynomial ring. If $f \in R$ is a Weierstrass polynomial then f is comaximal to every element in the multiplicative set $\mathcal{S} = R - (m, T)$.*

Proof. It is sufficient to show that (m, T) is the only maximal ideal in R that contains f. So let M be a maximal ideal in R that contains f. Since f is a monic polynomial, it follows that $M \cap A$ is a maximal ideal in A. As (A, m) is a local ring, we must have $M \cap A = m$. Thus $T \in M$ and hence $M = (m, T)$.

Remark 1.6. Let A be a ring and $\mathcal{S}$ be a multiplicative set in A. As customary, we denote by $A_{\mathcal{S}}$ the localization of A at $\mathcal{S}$. If $\mathcal{S} = A - p$ for a prime ideal p in A then we write $A_{\mathcal{S}} = A_p$.

PROPOSITION 1.7. *Let (A, m) be a local ring and let $R = A[T]$ be a polynomial ring. Suppose $f \in R$ is a Weierstrass polyomial then $R \subset R_{(m,T)}$ is an analytic isomorphism along f.*

Proof. Since the multiplicative set $\mathcal{S} = R - (m, T)$ in R contains no zero-divisors, we may regard R as a subring of $R_{(m,T)}$. Now for any h in $\mathcal{S}$, by Proposition 1.5, $(f, h) = R$. Therefore $1/h$ is in $R + fR_{(m,T)}$ and so we conclude that $R_{(m,T)} = R + fR_{(m,T)}$.

Suppose $a \in fR_{(m,T)} \cap R$. Then $a = f(g/h)$, where $g \in R$ and $h \in \mathcal{S}$. As $a \in R$ and $\mathcal{S}$ contains no zero-divisors in R, it follows that h divides fg in R. Then, since $(f, h) = R$, h must divide g in R. Hence $a \in fR$ and we are through.

In the rest of this section we investigate local rings and analytic isomorphisms therein.

DEFINITION 1.8. Let $B = A[X_1, \ldots, X_m]$ be a polynomial ring over a ring A. We shall say that a form F in B represents a unit in A if there exist elements $a_1, \ldots, a_m$ in A such that $F(a_1, \ldots, a_m)$ is a unit in A.

PROPOSITION 1.9. *Let (A, m) be a local ring and let $B = A[X_1, \ldots, X_m]$. Let M denote the the maximal ideal $(m, X_1, \ldots, X_m)$ of B. Given a form F in B that represents a unit in A. Then the local ring B_M contains a subring B_1 with the following properties*:

(1) *B_M is a localization of B_1 at a maximal ideal.*

(2) *B_1 is a polynomial ring in one variable over a local ring and B is contained in B_1.*

(3) *There is a monic polynomial f in $B_1 \cap FB_1$ such that $B_1 \subset B_M$ is an analytic isomorphism along f.*

Proof. Let $a_1,\dots, a_m$ be elements in A such that $F(a_1,\dots, a_m) = u$, where u is a unit in A. Since A is a local ring, at least one of the a_i's, say, a_1, must be unit in A. We apply the following homogeneous change of coordinates to B:

$$X_1 = a_1 Y_1, \qquad X_i = a_i Y_1 + Y_i \qquad \text{for} \quad i = 2,\dots, m.$$

Then $F(X_1,\dots, X_m) = F(a_1 Y_1, a_2 Y_1 + Y_2,\dots, a_m Y_1 + Y_m) = G(Y_1,\dots, Y_m)$ is a form in $B = A[Y_1,\dots, Y_m]$. Now $G(1, 0,\dots, 0) = F(a_1,\dots, a_m) = u$. Dividing G by u and calling the Y_i's as X_i's we have $G(X_1,\dots, X_m) = X_1^n + \alpha_1 X_1^{n-1} + \cdots + \alpha_n$, where n is the degree of G and each α_i is either zero or a form of degree i in the ring $A[X_2,\dots, X_m]$, for $i = 1, 2,\dots, n$.

We now construct B_1 with the required properties. Let $B' = A[X_2,\dots, X_m]$ and let $M' = (m, X_2,\dots, X_m)$. Set $B_1 = B'_{M'}[X_1]$. Obviously B_M is the localization of B_1 at the maximal ideal (M', X_1). As the multiplicative set $B_1 - (M', X_1)$ in B_1 contains no zero-divisors, B_1 is a subring of B_M. By similar reasoning B is a subring of B_1. Also B_1 is a polynomial ring over the local ring $B'_{M'}$. Finally, we set $f = G$. Then f becomes a Weierstrass polynomial in B_1. Hence, by Proposition 1.7, $B_1 \subset B_M$ is an analytic isomorphism along f. Thus properties (1), (2) and (3) hold for B_1.

Remark 1.10. Suppose J is an ideal in B that contains a form, say, F, representing a unit in A. Then we notice in the proof above that $G \in J$, and hence $f \in J$. This observation will be used later to apply Proposition 1.3.

PROPOSITION 1.11. *Let A be a ring and let $B = A[X_1,\dots, X_m]$. Then every ideal I of height $\geqslant 2$ in B contains a non-zero form in B.*

Proof. If some power of X_1 is in I then we are through. So we suppose that $\mathscr{S} \cap I = \phi$, where $\mathscr{S}$ is the multiplicative set $\{X_1^i\}_{i \geqslant 0}$ in B. Then $I_{\mathscr{S}}$ is an ideal of height $\geqslant 2$ in $B_{\mathscr{S}} = A[X_1,\dots, X_m, X_1^{-1}] = A[X_2/X_1,\dots, X_m/X_1][X_1, X_1^{-1}]$. Since $A[X_2/X_1,\dots, X_m/X_1][X_1]$ is a polynomial ring over $A[X_2/X_1,\dots, X_m/X_1] = B_1$ we conclude that the ideal $I_{\mathscr{S}} \cap B_1$ in B_1 has height $\geqslant 1$ and hence $I_{\mathscr{S}} \cap B_1 \neq (0)$. Let $0 \neq f \in I_{\mathscr{S}} \cap B_1$. If t is the total degree of f as a polynomial in B_1 then $X_1^t f = F$ is a form in B and $F \in I_{\mathscr{S}} \cap B$. This implies $X_1^i F \in I$ for some i as desired.

PROPOSITION 1.12. *Let $(A, (\pi))$ be a discrete valuation ring and let $B = A[X_1,\dots, X_m]$. Suppose I is an ideal in B with* $\mathrm{ht}(I) \geqslant 3$. *Then I contains a form in B outside πB.*

Proof. It is easy to see that we may assume I is a prime ideal in B with $\mathrm{ht}(I) \geqslant 3$. Now the following two cases arise.

Case I. $\pi \notin I$. By Proposition 1.11 I contains a form, say, F, in B. We write $F = \pi^t F_1$, with F_1 a form in B such that at least one coefficient of F_1 is a unit in A. Since I is a prime ideal in B and $\pi \notin I$, we have $F_1 \in I$. F_1 is a form in B outside πB.

Case II. $\pi \in I$. Let $k = A/(\pi)$ be the residue field of A. Let $\bar{I} = I/(\pi)$. Then $\bar{I}$ is an ideal in $k[X_1,..., X_m]$ with $\mathrm{ht}(\bar{I}) \geqslant 2$. By Proposition 1.11 $\bar{I}$ contains a non-zero form, say, $\bar{F}$, in $k[X_1,..., X_m]$. By lifting $\bar{F}$ to B we obtain a form in I outside πB.

Remark 1.13. As we will need it later, we notice that in the proof above if I is a prime ideal in B with $\mathrm{ht}(I) \geqslant 2$ and if $\pi \notin I$, then I contains a form in B outside πB.

Remark 1.14. Suppose $(A, (\pi))$ is a discrete valuation ring with infinite residue field. Then every form in $B = A[X_1,..., X_m]$ outside πB represents a unit in A.

We may as well cite that the requirement on the residue field to be infinite is necessary. For example, let $F_2 = \mathbb{Z}/(2)$ and $A = F_2[T]_{(T)}$. Then $f = XY(X+Y)$ is a form in $B = A[X, Y]$ outside TB. As $\bar{f}$ in $B/TB = F_2[X, Y]$ vanishes everywhere, f cannot represent a unit in A.

We conclude this section by recalling some facts about formal power series rings. The proof may be found in [7].

DEFINITION 1.15. Let $A = k[[X_1,..., X_d]]$ be a formal power series ring over a field k. A non-zero element f in A is called *regular of order* $s < \infty$ *in the variable* X_1 if X_1^s is the smallest power of X_1 (with non-zero coefficient in k) that occurs in f.

LEMMA 1.16. *Given a non-zero element f in $A = k[[X_1,..., X_d]]$, there is always a k-automorphism σ of A such that $\sigma(f)$ is regular (of some positive order) in X_1.*

THEOREM 1.17 (Weierstrass Preparation Theorem). *Let f be a non-zero, non-unit power series in $A = k[[X_1,..., X_d]]$. Suppose f is regular of order $s > 0$ in X_1. Then there exists a unique unit u in A such that $fu = X_1^s + \alpha_1 X_1^{s-1} + \cdots + \alpha_s$, where $\alpha_i \in k[[X_2,..., X_d]]$ and $\alpha_i(0, 0,..., 0) = 0$ for all i.*

Thus fu is a Weierstrass polynomial in $k[[X_2,..., X_d]][X_1]$.

PROPOSITION 1.8 (Weierstrass Division Theorem). *Let $f \neq 0$ in $A = k[[X_1,..., X_d]]$ be regular of order s in X_1. For any $g \in A$, there exist unique elements q, in A, and r, in the polynomial ring $k[[X_2,..., X_d]][X_1]$, such that $g = fq + r$, where either $r = 0$ or $\deg_{X_1} r < s$.*

We obtain the following important corollary.

COROLLARY 1.19. *Let* $f \neq 0 \in A = k[[X_1, \ldots, X_d]]$. *Suppose* f *is a Weierstrass polynomial in* $k[[X_2, \ldots, X_d]][X_1] = A_1$, *then* $A_1 \subset A$ *is an analytic isomorphism along* f.

Proof. If f is a Weierstrass polynomial in A_1 of degree, say, s then $f \in A$ is regular of order s in X_1. So, by Proposition 1.18, $A = A_1 + fA$. Now suppose that $g \in fA \cap A_1$. Since f is a monic polynomial in A_1, we can use divisibility in A_1 to write $g = fq + r$, where $q, r \in A_1$ and either $r = 0$ or degree of $r < s$. Moreover, $g = fq'$ for some $q' \in A$. Using uniqueness in Proposition 1.18 we conclude that $r = 0$ and $q' = q \in A_1$. Hence $g \in fA_1$ and so $A_1 \subset A$ is an analytic isomorphism along f.

2. THE MAIN RESULTS

In this section the main theorems of this paper are proved. We shall deal with polynomial rings with coefficients in a regular local ring. We organise the section in the following way. First we dispose of polynomial rings with coefficients in a formal power series ring over a field. Then we consider the cases where the coefficient rings are localizations of affine algebras. In principle, the method of proof in each of these theorems is quite similar. To avoid repeating the arguments, we start with an easy application of Mohan Kumar's theorem.

LEMMA 2.1. *Let* $R = A[T_1, \ldots, T_n]$, *where* A *is a ring of dimension* d. *Suppose* I *is an ideal in* R *such that* $\mathrm{ht}(I) > d$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$. *Then* I *is the homomorphic image of a projective* R*-module* P *of rank equal to* $\mu(I/I^2)$. *If in addition* A *is regular local then* $\mu(I) = \mu(I/I^2)$.

Proof. Since $\mathrm{ht}(I) > d = \dim(A)$, by [1, Section 4, Lemma 5], possibly after a change of variables, I contains a polynomial monic in one of the variables. So, by Theorem 1.2, I is the homomorphic image of a projective R-module P of rank equal to $\mu(I/I^2)$. Moreover, if A is regular local then P is extended from A as $\mathrm{rank}(P) > d$ [15] and hence free. Thus $\mu(I) = \mu(I/I^2)$.

THEOREM 2.2. *Let* $A = k[[X_1, \ldots, X_d]]$ *be a formal power series ring over a field* k *and let* $R = A[T_1, \ldots, T_n]$ *be a polynomial ring. If* I *is an ideal in* R *such that* $\mathrm{ht}(I) > \min\{d, n\}$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$, *then* $\mu(I) = \mu(I/I^2)$.

Proof. In view of Lemma 2.1, we may as well assume that $n < d$. Then $\mathrm{ht}(I) \geqslant n + 1$. It follows that $I \cap A \neq (0)$. If $0 \neq g \in I \cap A$, then, by Lemma 1.16 and Theorem 1.17, we can assume that g is a Weierstrass polynomial in $A_1 = k[[X_2, \ldots, X_d]][X_1]$. By Corollary 1.19, $A_1 \subset A$ is an analytic isomorphism along f. By letting $R_1 = A_1[T_1, \ldots, T_n]$ and $I_1 = I \cap R_1$, we observe that $R_1 \subset R$ is an analytic isomorphism along f and $f \in I_1$. Thus, by Proposition 1.3, it is enough to show that $\mu(I_1) = \mu(I_1/I_1^2)$.

Now the ideal I_1 contains the polynomial f monic in one of the variables. Moreover, $\mu(I_1/I_1^2) \geqslant \dim(I) + 2 = \dim(I_1) + 2$, by (1) in Proposition 1.3. By Theorem 1.2, there is a projective R_1-module of rank equal to $\mu(I_1/I_1^2)$ mapping onto I_1. Since finitely generated projective R_1-modules are free [11], we obtain $\mu(I_1) = \mu(I_1/I_1^2)$ as required.

As a corollary, in the case of one polynomial variable, we obtain the following:

COROLLARY 2.3. *Let $A = k[[X_1,..., X_d]]$ be a formal power series ring over a field k and let $R = A[T]$ be the polynomial ring in one variable. If I is an ideal in R with $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$.*

Proof. If $d = 0$, then $A = k$ is a field and there is nothing much to prove as R is a principal ideal domain. Therefore, we suppose that $d \geqslant 1$. Then $\min\{d, 1\} = 1$. Now for an ideal I in R with $\mathrm{ht}(I) \geqslant 2$, Theorem 2.2 is applicable. Thus we need consider the case of an ideal I in R with $\mathrm{ht}(I) = 1$. Since unmixed height 1 ideals in a unique factorization domain are principal, we may as well suppose that I is mixed. Now write $I = \lambda I'$ for some $\lambda \in R$, where I' is an ideal in R with $\mathrm{ht}(I') \geqslant 2$. It is obvious that $\mu(I) = \mu(I')$. Multiplication by λ gives an isomorphism of R-modules: $I'/I'^2 \xrightarrow{\approx} I/II'$. Also there is a surjective homomorphism of R-modules: $I/I^2 \to I/II'$. Hence it follows that $\mu(I'/I'^2) = \mu(I/II') \leqslant \mu(I/I^2)$. Now, since the assumption $\mu(I'/I'^2) < \mu(I/I^2)$ easily gives that $\mu(I) = \mu(I/I^2)$, we just consider the case $\mu(I'/I'^2) = \mu(I/I^2)$. By the hypothesis, $\mu(I'/I'^2) \geqslant \dim(I) + 2 \geqslant \dim(I') + 2$, and so, by Theorem 2.2, we conclude that $\mu(I') = \mu(I'/I'^2)$. Hence $\mu(I) = \mu(I/I^2)$.

The following proposition is used in the proofs of Theorem 2.5 and Proposition 3.1.

PROPOSITION 2.4. *Let (D, m) be a local ring and let $B = D[X_1,..., X_m]$ be a polynomial ring. Let M be the maximal ideal $(m, X_1,..., X_m)$ of B. Suppose I is an ideal in the polynomial ring $R = B_M[T_1,..., T_n]$ such that*

(i) *$\mu(I/I^2) \geqslant \dim(I) + 2$, and*

(ii) *$I \cap B$ contains a form in B that represents a unit in D. Then there is a projective R-module of rank equal to $\mu(I/I^2)$ mapping onto I.*

Proof. The ideal $J = I \cap B$ in B contains a form representing a unit in D. By Proposition 1.9, we obtain a subring B_1 of B_M satisfying (1), (2) and (3) therein. Also, by Remark 1.10 we may assume that $f \in J$. Now we let $R_1 = B_1[T_1,..., T_n]$ and $I_1 = I \cap R_1$. Then $R_1 \subset R$ is an analytic isomorphism along f and $f \in I_1$. Using (3) and (1) of Proposition 1.3, we have $\mu(I_1/I_1^2) =$

$\mu(I/I^2) \geqslant \dim(I) + 2 = \dim(I_1) + 2$. As the ideal I_1 in R_1 contains a polynomial, viz., f, monic in one of the variables, by Theorem 1.2 we get a projective R_1-module P of rank equal to $\mu(I_1/I_1^2)$ mapping onto I_1. Since R is a localization of R_1 (B is a localization of B_1) and $I_1R = I$, it follows that there is a projective R-module of the desired rank mapping onto I, as we wanted.

THEOREM 2.5. *Let K be an infinite field and let $C = K[X_1,..., X_d]$. Let M denote the maximal ideal $(f(X_1), X_2,..., X_d)$ of C, where f is a monic irreducible polynomial in $K[X_1]$. Let $A = C_M$ and set $R = A[T_1,..., T_n]$. Then:*

(1) *If I is an ideal in R with* $\mathrm{ht}(I) > \min\{d, n+2\}$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$, *then* $\mu(I) = \mu(I/I^2)$.

(2) *If I is a prime ideal in R with* $\mathrm{ht}(I) > \min\{d, n+1\}$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$, *then* $\mu(I) = \mu(I/I^2)$.

Proof. (1) If $\min\{d, n+2\} = d$ then $\mathrm{ht}(I) > d = \dim(A)$ and so we get the desired conclusion by Lemma 2.1. Thus we may assume that $\min\{d, n+2\} = n+2$. Then $\mathrm{ht}(I) \geqslant n+3$ and it follows that $\mathrm{ht}(I \cap A) \geqslant 3$. We look at A as: $A = D[X_2,..., X_d]_{(\pi, X_2,..., X_d)}$, where D is the discrete valuation ring $K[X_1]_{(f(X_1))}$ with uniformizing parameter $f(X_1) = \pi$. If $B = D[X_2,..., X_d]$, then it is easy to check that the contraction, J, of $I \cap A$ in B has the same height as $I \cap A$. Hence $\mathrm{ht}(J) \geqslant 3$. By Proposition 1.12, J contains a form, say, F, in B outside πB. Since the residue field $D/(\pi)$ is infinite we have, by Remark 1.14, that F represents a unit in D. Now Proposition 2.4 completes the proof as all finitely generated projective R-modules are free ([10] or [15]).

(2) As done in the proof of (1) we assume that $\min\{d, n+1\} = n+1$. Then $\mathrm{ht}(I) \geqslant n+2$. Since (1) is applicable for $\mathrm{ht}(I) > n+2$, we need only consider the case when $\mathrm{ht}(I) = n+2$. This gives that $\mathrm{ht}(I \cap A) \geqslant 2$. Now we consider the following two cases.

Case I. $f(X_1) \notin I \cap A$. As before, $A = D[X_2,..., X_d]_{(\pi, X_2,..., X_d)}$, where D is the discrete valuation ring $K[X_1]_{(f(X_1))}$ with uniformizing parameter $f(X_1) = \pi$, and $B = D[X_2,..., X_d]$. Since $I \cap A$ is a prime ideal in A of height $\geqslant 2$, $J = I \cap B$ is a prime ideal in B with $\mathrm{ht}(J) \geqslant 2$. Now $\pi \notin J$. By Remark 1.13, J contains a form in B outside πB. The rest of the argument is the same as that given in (1).

Case II. $f(X_1) \in I \cap A$. We now look at A as: $A = B[X_1]_{(M, f(X_1))}$, where $B = K[X_2,..., X_d]$ and M is the maximal ideal $(X_2,..., X_d)$ of B. Since $(M, f(X_1))$ is the only maximal ideal in $B_M[X_1]$ that contains $f(X_1)$, we have, by arguments similar to those given in the proof of Proposition 1.7, that $B_M[X_1] \subset A$ is an analytic isomorphism along f. By setting

$R_1 = B_M[X_1, T_1, \ldots, T_n]$ and $I_1 = I \cap R_1$, we obtain that $R_1 \subset R$ is an analytic isomorphism along f with $f \in I_1$. By Proposition 1.3, it is sufficient to show that $\mu(I_1) = \mu(I_1/I_1^2)$. We now note that the ideal I_1 contains a polynomial monic in one of the variables and we appeal to arguments similar to those given in the proof of Theorem 2.2 and so complete the proof.

Remark 2.6. We retain the notation of the previous theorem.

(a) If I is any ideal in R such that $\mu(I/I^2) \geqslant \dim(I) + 2$, we observe, in the proof of Case II above, that $\mu(I) = \mu(I/I^2)$ if $I \cap A$ contains $f(X_1)$ (in fact any power of $f(X_1)$). Moreover, if $I \cap A$ contains any power or any product of $X_2, \ldots, X_d$, then Proposition 2.4 can be used to conclude that $\mu(I) = \mu(I/I^2)$.

(b) We do not know if the restriction on the height of I in Theorem 2.5 (also in Theorem 2.2) is necessary.

One might even consider the following extension of Mohan Kumar's theorem.

Problem. Let A be a regular local ring of dimension d and let $R = A[T_1, \ldots, T_n]$ be a polynomial ring. If I is any ideal in R such that $\mu(I/I^2) \geqslant \dim(I) + 2$, is it true that $\mu(I) = \mu(I/I^2)$?

We know of no case where the desired equality fails. The conjecture due to Davis and Geramita mentioned in the Introduction is a special case of this problem.

We now shall use Theorem 2.5 to prove result (2) cited in the Introduction.

Let A be a local ring. We shall say that A is a *local algebra with a ground field* if A is a localization of a k-algebra of finite type for some field k. In this context k will be called a ground field for A. A local algebra with a ground field may very well possess more than one ground field as the following example shows.

EXAMPLE 2.7. Let $B = k[X, Y]$ be a polynomial ring over a field k and let $p = (X)$. Since $A = B_p = k[X, Y]_{(X)} = k(Y)[X]_{(X)}$, we see that k and $k(Y)$ are both ground fields for A.

Of special interest will be the situation when a regular local algebra A with a ground field possesses a ground field K such that the residue field of A is a finite separable and hence simple extension of K. If that is the case we shall refer to A as a *regular local algebra with a separating ground field.* For such a kind of algebras we prove the following theorem.

THEOREM 2.8. *Let A be a regular local algebra with a separating ground field K. Let* $\dim(A) = d$ *and let $\mathfrak{m}$ denote the maximal ideal of A.*

Pick an arbitrary element a in $\mathfrak{m}^2$. Then there exists a regular local subring S of A with the following properties:

(1) *S is a localization of a polynomial ring $C = K[X_1, ..., X_d]$ at a maximal ideal M of the type $(f(X_1), X_2, ..., X_d)$, where f is a monic irreducible polynomial in $K[X_1]$.*

(2) *There exists an element h in $S \cap aA$ such that $S \subset A$ is an analytic isomorphism along h.*

Proof. We choose elements $X_2, X_3, ..., X_d$ in $\mathfrak{m}$ such that $a, X_2, ..., X_d$ is a system of parameters in A and $X_2, ..., X_d$ are a part of a minimal generating set for $\mathfrak{m}$ (mod $\mathfrak{m}^2$). Since A is a regular local ring we have that $a, X_2, ..., X_d$ is a regular sequence in A. The field K is contained in A. Therefore $a, X_2, ..., X_d$ are algebraically independent over K. Thus $C_1 = K[a, X_2, ..., X_d]$ is a polynomial ring contained in A. Let B be the integral closure of C_1 in A and let $\mathfrak{m}_1 = \mathfrak{m} \cap B$. Since $\mathfrak{m}_1$ contracts to the maximal ideal $(a, X_2, ..., X_d)$ of C_1 and B is integral over C_1, it follows that $\mathfrak{m}_1$ is a maximal ideal in B.

Claim. $A = B_{\mathfrak{m}_1}$.

Proof. We first observe that A and B have the same fields of fractions. To see this, let frac(A) and frac(C_1) denote the fields of fractions of A and C_1, respectively. Both of these fields have the same transcendence degree, viz., d, over K. Since frac(C_1) is a subfield of frac(A), we must have that frac(A) is algebraic over frac(C_1). As B is the integral closure of C_1 in A, we conclude that the field of fractions of B is frac(A).

We now show that B is a Noetherian ring. In fact we show more by proving that B is a finite C_1-module. As A is a localization of a K-algebra of finite type, frac(A) is a finitely generated field extension of K a priori of frac(C_1). Hence frac(A) is a finite algebraic extension of frac(C_1). The ring C_1 is a pseudo-geometric ring [14, Theorem 36.5], therefore B is a finite C_1-module.

Now the normal local ring $B_{\mathfrak{m}_1}$ is analytically irreducible as the completion of $B_{\mathfrak{m}_1}$ is contained in the completion of A which is a domain. Further, the ideal $\mathfrak{m}_1 A$ in A is primary to $\mathfrak{m}$ and hence $A/\mathfrak{m}_1 A$ is a finite $B/\mathfrak{m}_1$-module. As $\dim(B_{\mathfrak{m}_1}) = \dim(A)$, by Zariski' main theorem [14, Theorem 37.4] we conclude that $A = B_{\mathfrak{m}_1}$ and the claim is proved.

In order to simplify the forthcoming notation, let us refer to A as $B_{\mathfrak{m}_1}$ and we may as well rename $\mathfrak{m}_1$ as $\mathfrak{m}$.

By the given hypothesis, $L =$ residue field of $A = B/\mathfrak{m} = K(\bar{\alpha})$ for some α in B. Let f denote the minimal polynomial of $\bar{\alpha}$ over K. Then $f(\alpha) \in \mathfrak{m}$. As L is a separable extension of K we get that $f'(\alpha) \notin \mathfrak{m}$.

Claim. We can choose $X_1 = \alpha$ in such a way that $(a, f(X_1), X_2, ..., X_d) = \mathfrak{m}$.

Proof. Note that $\mathfrak{m}$ is a regular maximal ideal in B and $\mathrm{ht}(\mathfrak{m}) = d$. The elements $X_2, \ldots, X_d$ are a part of a minimal generating set for $\mathfrak{m}$ (mod $\mathfrak{m}^2$). It is easy to see that we can replace α by $\alpha + Y$ for some suitable Y in $\mathfrak{m}$ if necessary, to assume that $f(\alpha), X_2, \ldots, X_d$ is a minimal generating set for $\mathfrak{m}$ (mod $\mathfrak{m}^2$). Hence $f(\alpha), X_2, \ldots, X_d$ generate $\mathfrak{m}$ in $A = B_{\mathfrak{m}}$.

Now there may be several (but finite in number) maximal ideals in B that lie over the maximal ideal $(a, X_2, \ldots, X_d)$ of C_1. Let them be $\mathfrak{m}_1 = \mathfrak{m}, \mathfrak{m}_2, \ldots, \mathfrak{m}_r$. Choose X_1 in B such that $X_1 \equiv \alpha (\mathrm{mod}\ \mathfrak{m}^2)$ and $X_1 \equiv 0$ (mod $\mathfrak{m}_i$), for $i = 2, \ldots, r$. Then the ideal generated by $a, f(X_1), X_2, \ldots, X_d$ in B is contained in $\mathfrak{m}$ only. Since $f(X_1) \equiv f(\alpha)$ (mod $\mathfrak{m}^2$), $f(X_1), X_2, \ldots, X_d$ generate $\mathfrak{m}$ in A. It now follows that $\mathfrak{m} = (a, f(X_1), \ldots, X_d)$ and the claim is proved.

Further, as X_1 is integral over $K[a, X_2, \ldots, X_d]$, replacing X_1 by $X + a^j$ for some integer j large enough, we may even assume that a is integral over $K[X_1, \ldots, X_d]$. We now define $C = K[X_1, \ldots, X_d]$ and $M = (f(X_1), X_2, \ldots, X_d)$. Set $S = C_M$. Note that $\mathfrak{m} \cap C = M$. We look at the ring $C[a]$ and the maximal ideal (M, a) of $C[a]$. As (M, a) generates $\mathfrak{m}$ in B, B is a finite $C[a]$-module and $B/\mathfrak{m} = L \simeq C[a]/(M, a)$, we conclude, using Nakayama's lemma, that $A = B_{\mathfrak{m}} = C[a]_{(M,a)}$.

Since $A/aA = C[a]_{(M,a)}/aC[a]_{(M,a)} = S/S \cap aA$, we get $A = S + aA$. We define a C-algebra homomorphism σ: $C[T] \to C[a]$ by $\sigma(T) = a$. As a is integral over C, there is an irreducible monic polynomial in $C[T]$,

$$F(T) = T^n + \lambda_{n-1} T^{n-1} + \cdots + \lambda_1 T + \lambda_0,$$

such that $\sigma(F(T)) = F(a) = 0$. Thus $\lambda_0 = -a(\lambda_1 + \lambda_2 a + \cdots + \lambda_{n-1} a^{n-2})$. If we knew that $\lambda_1 \notin M$, then $\lambda_1 + \lambda_2 a + \cdots + \lambda_{n-1} a^{n-2} \notin (M, a)$ and hence $aA = \lambda_0 A$. By setting $h = \lambda_0$ we would have that $A = S + hA$ and $h \in S \cap aA$. To see $\lambda_1 \notin M$, we observe that $A = C[a]_{(M,a)}$ and M generates the maximal ideal of A.

Finally, in order to conclude that $S \cap A$ is an analytic isomorphism along h we must show that $S \cap hA = hS$. For this it is enough to remark that A is a faithfully flat S-algebra [12, 4.c]. Observe that $S[a]$ is a free S-module of finite rank and hence $S[a]$ is a flat S-module. Since A is a localization of $S[a]$ (at the maximal ideal (M, a)), A is a flat $S[a]$-module. By transitivity of flatness [12, 3.b], A is a flat S-module. As S is a local ring, A is a faithfully flat S-algebra and we are through.

Now we are in a position to use Theorem 2.5 and prove:

THEOREM 2.9. *Let A be a d-dimensional regular local algebra with a separating ground field K. Let $R = A[T_1, \ldots, T_n]$ be a polynomial ring. Suppose that the residue field of A is infinite. Then*:

(1) *If I is an ideal in R with* $\mathrm{ht}(I) > \min\{d, n+2\}$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$, *then* $\mu(I) = \mu(I/I^2)$.

(2) *If I is a prime ideal in R such that* $\mathrm{ht}(I) > \min\{d, n+1\}$ *and* $\mu(I/I^2) \geqslant \dim(I) + 2$, *then* $\mu(I) = \mu(I/I^2)$.

Proof. We reduce the proof to the case of Theorem 2.5. If $I' = I \cap A$ then I' is a non-zero ideal in A. With the help of Theorem 2.8, we can get a local subring S of A such that S is a localization of the polynomial ring $K[X_1,..., X_d]$ at a maximal ideal of the type $(f(X_1), X_2,..., X_d)$, with $f(X_1)$ an irreducible monic polynomial in $K[X_1]$ and that $S \subset A$ is an analytic isomorphism along h for some element h in $I' \cap S$. We set $R_1 = S[T_1,..., T_n]$ and $I_1 = I \cap R_1$. Then $R_1 \subset R$ is an analytic isomorphism along h and $h \in I_1$. By Proposition 1.3, $\mu(I_1/I_1^2) = \mu(I/I^2)$. If we knew that $\mathrm{ht}(I_1) = \mathrm{ht}(I)$, then $\mathrm{ht}(I_1) > \min\{d, n+2\}$ or $\mathrm{ht}(I_1) > \min\{d, n+1\}$, as the case would be for $\mathrm{ht}(I)$, and so we would be in the situation of Theorem 2.5. Thus, we need only show that $\mathrm{ht}(I_1) = \mathrm{ht}(I)$.

We have $R/hR \simeq R_1/hR_1$ and $I/hR \simeq I_1/hR_1$ (as R/hR or R_1/hR_1-modules). Thus we can identify the rings R/hR and R_1/hR_1. So the ideal I/hR in R/hR can be identified with the ideal I_1/hR_1 in R_1/hR_1. Therefore, $\mathrm{ht}(I_1/hR_1) = \mathrm{ht}(I/hR)$. Moreover, hR_1 and hR are principal ideals in unique factorization domains, and we conclude that $\mathrm{ht}(I_1) = \mathrm{ht}(I_1/hR_1) + 1 = \mathrm{ht}(I/hR) + 1 = \mathrm{ht}(I)$, and so we are through.

The following lemma is due to Lindel [9]. It gives a sufficient condition for a local domain to be a local algebra with a separating ground field.

LEMMA 2.10. *Let A be a local domain. Suppose that A is a local algebra with a ground field k and that k is a finitely generated field extension of a perfect field k_0. Then A is a local algebra with a separating ground field.*

Proof. Since the local domain A is a localization of an affine k-algebra and k is a finitely generated extension of k_0, it turns out that A is a localization of an affine k_0-domain. So we choose an affine k_0-domain $B = k_0[t_1,..., t_n]$ and a prime ideal p in B such that $A = B_p$. The residue field L of A equals $k_0(x_1,..., x_n)$, where $x_i \equiv t_i \pmod{p}$, for $i = 1, 2,..., n$. Since k_0 is a perfect field and L over k_0 is finitely generated, by [16, Vol. I, p. 105, Theorem 31] L is separably generated over k_0. Let $Tr.\, d\, L/K_0 = r$. By [16, Vol. I, p. 104, Theorem 30], we may even assume that $x_1, x_2,..., x_r$ is a separating transcendence basis of L over k_0. It then follows that $k_0[t_1,..., t^r]$ is a polynomial subring of B and $k_0[t_1,..., t_r] \cap p = (0)$. Let $K = k_0(t_1,..., t_r)$. Now $A = C_q$, where $C = K[t_{r+1},..., t_n]$ and $q = pC$. As L is a finite separable extension of K, we obtain that K is a separating ground field for A.

We do not know whether every local algebra with a ground field is a local algebra with a separating ground field.

3. Some Applications

As mentioned in the Introduction in this section we give a few applications of the techniques and the results proved in the previous sections.

Let R be a ring. An ideal I in R is called a *weak complete intersection* if I can be generated by $\mathrm{ht}(I)$ elements. If I can be generated by a regular sequence (necessarily of length equal to $\mathrm{ht}(I)$) then I is called a *complete intersection*. In case R is a Cohen–Macaulay ring both these notions coincide [5, 11.19, 11.11].

We start with result (3) cited in the Introduction. We may as well state this in a slightly general form.

Proposition 3.1. *Let $(D, (\pi))$ be a discrete valuation ring with infinite residue field and let $B = D[X_1,..., X_{d-1}]$ be a polynomial ring. Let $A = B_{(\pi, X_1,\ldots,X_{d-1})}$, $(\dim(A) = d)$ and set $R = A[T_1,..., T_n]$. Then*:

(1) *If I is an ideal in R with $\mathrm{ht}(I) > \min\{d, n+2\}$ and $\mu(I/I^2) \geqslant \dim(I) + 2$, then $\mu(I) = \mu(I/I^2)$.*

(2) *Every maximal ideal in R is a complete intersection.*

Proof. We first note that all finitely generated R-modules are free. This can be seen as follows. A well-known result of Quillen and Suslin [7, p. 134, Theorem 2.7] says that all finitely generated projective modules over $B[T_1,..., T_n] = D[X_1,..., X_{d-1}, T_1,..., T_n]$ are free, and hence extended from B. We now use Roitman [15, Proposition 2] to conclude that every finitely generated projective R-module is extended from A and hence free.

(1) This case can be treated exactly the same way as (1) in Theorem 2.5 was proved and so we leave it.

(2) Let M be any maximal ideal in R. By [4], we may even assume that $d \geqslant 3$. Since M is a maximal ideal in a polynomial ring in n variables over a local ring of dimension d, it follows that $\mathrm{ht}(M)$ is either $d+n$ or $d+n-1$. If $n \geqslant 2$, then $\mathrm{ht}(M) > d \geqslant \min\{d, n+2\}$ and so we conclude that $\mu(M) = \mu(M/M^2) = \mathrm{ht}(M)$ (M is a regular maximal ideal). Therefore, we may suppose that $n = 1$. Then, since $\mathrm{ht}(M) \geqslant d \geqslant 3 = \min\{d, n+2\}$, we just consider the situation $\mathrm{ht}(M) = d = 3$ otherwise (1) would be applicable. We consider the following two possibilities:

(i) $\pi \in M$. Let $\bar{M} = M/(\pi)$. Then $\bar{M}$ is a maximal ideal in $\bar{R} = R/(\pi) = \bar{A}[T]$, where $\bar{A} = A/(\pi)$ is a regular local ring of dimension 2. By [4], $\mu(\bar{M}) = \mathrm{ht}(\bar{M}) = 2$. Hence $\mu(M) = 3$ as desired.

(ii) $\pi \notin M$. We now can follow the same argument as that given in Case I of (2) in Theorem 2.5 and so complete the proof.

PROPOSITION 3.2. *Let $A = k[[X_1,..., X_d]]$ be a power series ring over a field and let $R = A[T_1,..., T_n]$ be a polynomial ring. Suppose I is an ideal in R such that I/I^2 is a free R/I-module or rank $\geqslant \dim(I) + 2$. Then I is a complete intersection. In particular, every maximal ideal in R is a complete intersection.*

Proof. As I is an ideal in R such that I/I^2 is a free R/I-module, it follows that rank $(I/I^2) = \mathrm{ht}(I)$, say, t [6, Lemma 1]. By hypothesis, $t \geqslant \dim(I) + 2$. Moreover, as R is a catenarian ring and every maximal ideal in R has height at least $\dim(R) - 1$, it follows that $\dim(I) \geqslant \dim(R) - 1 - t$. Thus, $t \geqslant d + n - 1 - t + 2$ or $2t \geqslant d + n + 1$. Now it is obvious that $t > \min\{d, n\}$ and hence, by Theorem 2.2, $\mu(I) = \mu(I/I^2) = t$. Since R is Cohen–Macaulay (in fact regular) I is a complete intersection.

PROPOSITION 3.3. *Let (A, m) be an equi-characteristic regular local ring of dimension d. Let $R = A[T_1,..., T_n]$ be a polynomial ring over A. If $(\hat{A}, \hat{m})$ denotes the m-adic completion of (A, m), then let $R^* = \hat{A}[T_1,..., T_n]$. For an ideal I in R, let $I^* = IR^*$. Suppose I is an ideal in R such that I/I^2 is a free R/I-module of rank $\geqslant \dim(I) + 2$, then I^* is a complete intersection in R^*.*

Proof. Since $(\hat{A}, \hat{m})$ is an equi-characteristic complete regular local ring of dimension d, $\hat{A} = k[[X_1,..., X_d]]$, where $k \simeq A/m$ [16, Vol. II, p. 307]. Since $I^* = IR^*$ and the rings R and R^* are Cohen–Macauley and also R^* is a flat extension of R [12, 3.c], we conclude that $\mathrm{ht}(I) \leqslant \mathrm{ht}(I^*)$. Also, as I/I^2 is a free R/I-module, say, of rank t, it follows that I is an ideal of height t in R. Further, it is easy to see that $\mu(I^*/I^{*2}) \leqslant \mu(I/I^2)$, and so $\mu(I^*/I^{*2}) \leqslant t \leqslant \mathrm{ht}(I^*)$. But $\mu(I^*/I^{*2}) \geqslant \mathrm{ht}(I^*)$, hence we conclude that $\mu(I^*/I^{*2}) = \mathrm{ht}(I^*) = t = \mu(I/I^2)$.

Claim. $\mu(I^*/I^{*2}) \geqslant \dim(I^*) + 2$.

Proof. As $\mu(I/I^2) \geqslant \dim(I) + 2$, it is enough to show that $\dim(I) \geqslant \dim(I^*)$. Now $\dim(I)$ is either $d + n - t$ or $d + n - t - 1$. If $\dim(I) = d + n - t$, then $\dim(I) = d + n - t = \dim(R^*) - \mathrm{ht}(I^*) \geqslant \dim(I^*)$. So we consider the possibility $\dim(I) = d + n - t - 1$. This happens only when I is contained in no maximal ideal of height $d + n$ in R^*, as it is easy to verify that a maximal ideal of height $d + n$ in R^* contracts to a maximal ideal of height $d + n$ in R. Thus, $\dim(I) = d + n - t - 1 \geqslant \dim(I^*)$ and the claim is proved.

Now I^* is an ideal in a polynomial ring over a power series ring, and $\mu(I^*/I^{*2}) = \mathrm{ht}(I^*) \geqslant \dim(I^*) + 2$, we appeal to the argument given in the proof of Proposition 3.2 (in fact I^*/I^{*2} is a free R^*/I^*-module of rank t) to conclude that $\mu(I^*) = \mu(I^*/I^{*2})$. As R^* is Cohen–Macaulay, I^* is a complete intersection.

Remark 3.4. It would be quite interesting to know if I itself is a complete intersection in R.

Finally, we state

PROPOSITION 3.5. *Let A be a d-dimensional regular local algebra with a separating ground field. Suppose that the residue field of A is infinite. Then every maximal ideal in $R = A[T_1, \ldots, T_n]$ is a complete intersection.*

Proof. This follows from Theorem 2.9.

ACKNOWLEDGMENTS

The results of this paper were obtained in my Ph.D. dissertation written at Queen's University, Kingston, Canada. I am grateful to my teacher, Professor A. V. Geramita, for the mathematical training and the moral support I received from him. Also I owe my sincere thanks to Professor E. D. Davis at Albany for many helpful discussions. The ring-theoretic content of the Zariski's main theorem on birational transformations was made clear to me by Professor Davis. I thank Professors Leslie Roberts and Chuck Weibel for several useful criticisms. I am also grateful to Dr. Paul Eakin for several useful suggestions.

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